Solutions: Homework 7

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Problem 1. Let $G = \mathbb{C} \setminus \{0\}$ and show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.

Proof. Let $\gamma : [0,1] \to G$ be a closed curve in G. Let $\gamma' : [0,1] \to \{z : |z| = 1\}$ be the curve given by

$$\gamma'(s) = \frac{\gamma(s)}{|\gamma(s)|}.$$

This is well-defined because $\gamma(s) \neq 0$ for all $s \in [0,1]$. We will show that γ and γ' are homotopic. Define $\Gamma : [0,1] \times [0,1] \to G$ by

$$\Gamma(s,t) = (1-t)\gamma(s) + t\gamma'(s)$$

Note that $\Gamma(s,t) \neq 0$ for all s,t and so it is well-defined. It is clearly continuous and

$$\Gamma(s,0) = \gamma(s), \quad \Gamma(s,1) = \gamma'(s)$$

$$\Gamma(0,t) = \Gamma(1,t) \quad (0 \leq t \leq 1).$$

So γ is homotopic to γ' in G. □

Problem 2. Let $G = \mathbb{C} \setminus \{a,b\}, a \neq b$, and let γ be the curve in the book. Show that $n(\gamma; a) = n(\gamma; b) = 0$.

Proof. This solution is not rigorous. We see that there are two closed curves going around a with one going in the clockwise direction and the other in the anti-clockwise direction. This means that the index contributed by one of them is 1 and the other one is -1. Adding up, we see that $n(\gamma; a) = 0$. The same argument holds for b. □

Problem 3. Let G be a region and let γ_0 and γ_1 be two closed smooth curves in G. Suppose $\gamma_0 \sim \gamma_1$ and Γ satisfies (6.2). Also suppose that $\gamma_t(s) = \Gamma(s,t)$ is smooth for each t. If $w \in \mathbb{C} \setminus G$ define $h(t) = n(\gamma_t; w)$ and show that $h : [0,1] \to \mathbb{Z}$ is continuous.

Proof. Since $[0,1]$ is connected, this is equivalent to showing that h is constant. We know that, by Cauchy's theorem, if γ and γ' are two homotopic closed rectifiable curves in G, then $n(\gamma; w) = n(\gamma'; w)$ for all $w \in \mathbb{C} \setminus G$. We will prove that for all $t \in [0,1], \gamma_0$ is homotopic to γ_t. This shows that $h(0) = h(t)$ for all $t \in [0,1]$, and hence h is constant. Fix $0 \leq t_0 \leq 1$. Let $\Gamma' : [0,1] \times [0,1] \to G$ by

$$\Gamma'(s,t) = \Gamma(s, t_0 t)$$

Then Γ' is a homotopy from γ_0 to γ_{t_0}. This concludes our proof. □

Problem 4. Let G be open and suppose that γ is a closed rectifiable curve in G such that $\gamma \approx 0$. Set $r = d(\{\gamma\}, \partial G)$ and $H = \{z \in \mathbb{C} : n(\gamma; z) = 0\}$.
(a) Show that $\{z : d(z, \partial G) < \frac{1}{2}r\} \subset H$.
(b) Use part (a) to show that if $f : G \to \mathbb{C}$ is analytic then $f(z) = \alpha$ has at most a finite number of solutions z such that $n(\gamma; z) \neq 0$.

Proof. (a) Let z be such that $d(z, \partial G) < \frac{1}{2}r$. Then there exists $x \in \partial G$ such that $d(z, x) < \frac{1}{2}r$. Then $B(x; \frac{1}{2}r)$ is a connected subset of $\mathbb{C} \setminus \{\gamma\}$. Then $n(\gamma; .)$ is constant on $B(x; \frac{1}{2}r)$. But $B(x; \frac{1}{2}r) \cap (\mathbb{C} \setminus G) \neq \emptyset$. Since $\gamma \approx 0$, this shows that $n(\gamma; z) = 0$. As z is arbitrary, this completes the proof.
(b) WLOG, assume that $\alpha = 0$. Assume that f is not the constant function. Let $Z = \{z \in G : f(z) = 0\}$. Then Z has no limit points in G, by Theorem 3.7. This implies that any limit point lies in ∂G. Now we know that the set $\{z \in \mathbb{C} : n(\gamma; z) \neq 0\}$ is bounded. Suppose there exists infinitely many $z \in G$ such that $f(z) = 0$ and $n(\gamma; z) \neq 0$, and denote the set of all such z by V. The set $\{z \in \mathbb{C} : n(\gamma; z) \neq 0\}$ is bounded. So V is bounded, hence $\overline{V}$ is compact. So there exists a sequence $\{x_n\}$ in V that converges to x in $\overline{V}$. But we know that Z and hence V has no limit points in G. So $x \in \partial G$. Then, by continuity, $n(\gamma; x) \neq 0$, which contradicts (a). ☐

Problem 5. Let f be analytic in $B(a; R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m iff $f^{(m-1)}(a) = ... = f(a) = 0$ and $f^{(m)}(a) \neq 0$.

Proof. Suppose that a is a zero of multiplicity m. Then there exists an analytic function $g : B(a; R) \to \mathbb{C}$ such that $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$. Then, $h(z) = (z-a)^{m-1} g(z)$ has a zero of multiplicity $m-1$ at a. Inductively, we assume that $h^{(m-2)}(a) = ... = h(a) = 0$ and $h^{(m-1)}(a) \neq 0. f(z) = (z-a)h(z)$. So $f^{(i)}(z) = (z-a)h^{(i)}(z) + \sum_{j=0}^{i-1} h^{(j)}(z)$. Then we see that $f^{(m-1)}(a) = ... = f(a) = 0$ and $f^{(m)}(a) \neq 0$.
Conversely, suppose $f^{(m-1)}(a) = ... = f(a) = 0$ and $f^{(m)}(a) \neq 0$. Let a be a zero of multiplicity k. Then $f^{(k)}(a) \neq 0$, hence $k \geq m$, but $f^{(i)}(a) = 0$ for $i < k$ by the above paragraph. This implies that $k \leq m$. So $k = m$. ☐

Problem 6. Suppose that $f : G \to \mathbb{C}$ is analytic and one-one; show that $f'(z) \neq 0$ for any z in G.

Proof. Suppose $f'(a) = 0$ for some $a \in G$. Let $g : G \to \mathbb{C}$ be defined as $g(z) = f(z) - f(a)$. Then $g(a) = g'(a) = 0$. So g has a zero at a of multiplicity at least 2, say, m. Then, by Theorem 7.4, there exists $\epsilon > 0$ and $\delta > 0$ such that for $0 < |\zeta| < \delta$, the equation $g(z) = \zeta$ has exactly m simple roots in $B(a; \epsilon)$. This contradicts the fact that g is one-one. ☐